Quantitative Causality in Membrane Systems

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What is Causality?

When looking at a transition $u \xrightarrow{F} u'$ and look at a part (submultiset) v of u', we would like to be able to say it is caused by a certain part of u and a certain part of F.

$$u \xrightarrow{F} u' \geq v$$

$$lhs(F) + (u - lhs(F)) \xrightarrow{F} v + (u' - v)$$

We look for $G \leq F$, $w \leq u - lhs(F)$ minimal such that $rhs(G) + w \geq v$.

$$\color{red}{lhs(G) + w} + \ldots = u \xrightarrow{F} \color{red}{v} + \ldots = u'$$

- cover of v: produces v (and possibly some other objects)
- cause of v: minimal cover
- local cover and cause: defined in relation to a transition in which v is produced
- (global) cover and cause: defined generally

General Notions:

For multisets:

- $(x \cap y)(a) = min\{x(a), y(a)\}$;
- $(x \cup y)(a) = max\{x(a), y(a)\}$;
- $(x \setminus y)(a) = max\{x(a) - y(a), 0\}$.

For rules:

- w is called *stable* if there is no rule $r \in R$ which can be applied to w: $\nexists r \in R$ such that $lhs(r) \leq w$;
- F is called *maximally parallel* with respect to u if $lhs(F) \leq u$ and $u - lhs(F)$ is stable;
- u is said to evolve to u' through F (denoted by $u \xrightarrow{F} u'$) whenever F is maximally parallel with respect to u and $u' = u - lhs(F) + rhs(F)$.

Local Causality

$$u \xrightarrow{F} u' \geq v$$

Definition (Local cover)

$(G, w) \in Cover_{u,F}(v)$ whenever:

- F is maximally parallel with respect to u and $G \leq F$;
- $w \leq v$ and $w \leq u - lhs(F)$;
- $v \leq w + rhs(G)$.

Example

$r_1 : x \to a + b$, $r_2 : y \to b$

$$u = 2x + y + a + b \xrightarrow{F=2r_1+r_2} u' = 3a + 4b \geq v = 2a$$

Local covers of v are $(2r_1, a), (r_1 + r_2, a), (2r_1, 0), (r_1, a) \ldots$. Minimal covers are $(2r_1, 0)$ and (r_1, a).

Local Causality

If (G, w) is a minimal (local) cover then $w = v \backslash rhs(G)$.

G is called a local cause of v whenever $(G, v \backslash rhs(G))$ is a minimal local cover of v.

Example

$r_1 : x \to a + b$, $r_2 : y \to b$

$$u = 2x + y + a + b \xrightarrow{F = 2r_1 + r_2} u' = 3a + 4b \geq v = 2a$$

Minimal local covers of v are $(r_1, a), (2r_1, 0)$. Local causes are r_1 and $2r_1$.

Example

$r_1 : x \to a + b$, $r_2 : y \to b$

$$u = 2x + y \xrightarrow{F = 2r_1 + r_2} u' = 2a + 3b \geq v = 2a$$

Minimal local cover of v is only $(2r_1, 0)$, thus there is only one local cause: $2r_1$.

Global Causality

Covers and causes can be defined independently of a certain transition.

Definition

$(G, w) \in Cover(v)$ whenever w stable and $w \leq v \leq w + rhs(G)$.
$G \in Cause(v)$ if there is a w such that (G, w) is a minimal cover of v.

Proposition

$$Cover(v) = \bigcup_{(u,F)\in\mathcal{F}} Cover_{u,F}(v) \quad and \quad Cause(v) = \bigcup_{(u,F)\in\mathcal{F}} Cause_{u,F}(v).$$

where $\mathcal{F}$ denotes the family of pairs (u, F) for which $u \xrightarrow{F} u' \geq v$.

Caracterisation Theorem

Let v obtainable from u through F and $G \leq F$. Then $G \in Cause_{u,F}(v)$ iff:

- $v \backslash rhs(G) \leq u - lhs(F)$;
- $rhs(G) \cap v > rhs(G - r) \cap v$ for any $r \in G$.

Consequently, $G \in Cause(v)$ iff:

- $v \backslash rhs(G)$ stable;
- $rhs(G) \cap v > rhs(G - r) \cap v$ for any $r \in G$.

This yields an inductive construction for $Cause(v)$. Let $\mathcal{C}(v)$ be the smallest set with the properties:

- $(0, v) \in \mathcal{C}(v)$;
- $(G, w) \in \mathcal{C}(v)$ and r such that $rhs(r) \cap w > 0$, then $(G + r, w \backslash rhs(r)) \in \mathcal{C}(v)$.

Let $\bar{\mathcal{C}}(v)$ be the set of elements (G, w) of $\mathcal{C}(v)$ for which w is stable. Then

$$Cause(v) = \left\{ G \mid \exists w \text{ such that } (G, w) \text{ minimal element of } \bar{\mathcal{C}}(v) \right\}.$$

Examples

Consider the rules $r_1 : x \to a + b$, $r_2 : y \to b$ and $v = 2a$. Then:

- $(0, 2a) \in \mathcal{C}(v)$;
- $(0 + r_1, 2a \backslash rhs(r_1) = (r_1, a) \in \mathcal{C}(v)$;
- $(r_1 + r_1, a \backslash rhs(r_1) = (2r_1, 0) \in \mathcal{C}(v)$;
- $0, a, 2a$ stable so $Cause(v) = \{0, r_1, 2r_1\}$.

Let $v' = a + b$. Then:

- $(0, a + b) \in \mathcal{C}(v')$;
- $(0 + r_1, a + b \backslash rhs(r_1)) = (r_1, 0) \in \mathcal{C}(v')$;
- $(0 + r_2, a + b \backslash rhs(r_2)) = (r_2, a) \in \mathcal{C}(v')$;
- $(r_2 + r_1, a \backslash rhs(r_1)) = (r_1 + r_2, 0) \in \mathcal{C}(v')$
- thus $\bar{\mathcal{C}}(v') = \{(0, a + b), (r_1, 0), (r_2, a), (r_1 + r_2, 0)\}$ and $Cause(v') = \{0, r_1, r_2\}$.

Examples

Consider the rules $r_1 : x \to a + b$, $r_2 : y \to b$, $r_3 : a + b \to y$ and $v' = a + b$. Then:

- $(0, a + b) \in \mathcal{C}(v')$;
- $(0 + r_1, a + b \backslash rhs(r_1)) = (r_1, 0) \in \mathcal{C}(v')$;
- $(0 + r_2, a + b \backslash rhs(r_2)) = (r_2, a) \in \mathcal{C}(v')$;
- $((r_2 + r_1, a \backslash rhs(r_1)) = (r_1 + r_2, 0) \in \mathcal{C}(v')$
- $a + b$ is not stable thus $\bar{\mathcal{C}}(v') = \{(r_1, 0), (r_2, a), (r_1 + r_2, 0)\}$ and $Cause(v') = \{r_1, r_2\}$.

Having 0 as a cause would mean that v' can be entirely obtained from objects a, b which are not evolved in an evolution step. This does not take place since if a and b were both in a multiset u, when u evolved $a + b$ would be consumed by rule r_3.

Causal Characterisation of u

Theorem

Let u, v be multisets of objects. Then $\exists u \xrightarrow{F} u'$ such that $u' \geq v$ if and only if there exist $G \in Cause(v)$, H a multiset of rules and w stable such that $w \geq v \backslash rhs(G)$ and $u = lhs(G) + lhs(H) + w$.

Example

Consider rules $r_1 : x \to a + b$, $r_2 : y \to b$, $r_3 : a + b \to y$ and $v = a + b$.
Then $Cause(v) = \{r_1, r_2\}$.
Let $H = \alpha \cdot r_1 + \beta \cdot r_2 + \gamma \cdot r_3$.
For $G = r_1$, w stable then $w = \delta \cdot a$ or $w = \omega \cdot b$.
Then either $u = x + \alpha \cdot x + \beta \cdot y + \gamma \cdot (a + b) + \delta \cdot a$ or $u = x + \alpha \cdot x + \beta \cdot y + \gamma \cdot (a + b) + \omega \cdot b$.
Thus u is characterized by $u(x) \geq 1$.
For $G = r_2$, w is stable and $w \geq a$ so $w = \delta \cdot a$, with $\delta \geq 1$.
Then $u = y + \alpha \cdot x + \beta \cdot y + \gamma \cdot (a + b) + \delta \cdot a$, which translates to $u(y) \geq 1$ and $u(a) \geq u(b) + 1$.
Thus $a + b$ can be obtained from u iff $u(x) \geq 1$ or $u(y) \geq 1, u(a) \geq u(b) + 1$.

Conclusion

- we introduce and study quantitative causality (based on objects)
- local/global notions of causality (covers $\rightarrow$ causes)
- inductive construction for obtaining global causes
- global causes $\rightarrow$ static characterisation of u such that v can be obtained from u

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Thank you for listening!